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Long-time behavior of the semiclassical baker's map

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We study the long-time behavior of the quantized baker's map in the semiclassical approximation. Our main object of investigation is the trace of the (n-step) time-evolution operator. We express this trace as a phase-space integral which equals the semiclassical expression in terms of periodic orbits. This enables us to follow the evolution explicitly up to the time at which the semiclassical traces start to diverge exponentially from the quantum ones. Our data indicate that this breakdown time scales with h in a way close to $h^{-1/2}$.

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During the past few years there has been an increasing interest in the range of validity of the semiclassical approximation (SCA). From general considerations about the correspondence between quantum and classical quantities one expects that, for any given value of Planck's constant h, the SCA should correctly describe the quantum dynamics for times shorter than a "breakdown" time $t_B(h)$ which increases as h decreases. The knowledge of the *specific* h dependence of t_B is of central importance for determining whether it is possible to resolve individual eigenenergies in the limit $h \to 0$. For two-dimensional systems, the mean level distance scales as h^2 and, consequently, a time of the order $1/h$ is required to do this. For maps with a two-dimensional phase space quasienergies are separated on average by $2\pi h$, and these can be resolved, again, if the breakdown time is proportional to $1/h$.

The starting point for the SCA in the time domain is the semiclassical propagator [1, 2]. Using it, Berry and co-workers [3, 4] (see also [5]) investigated the formation of phase-space structures for chaotic systems and concluded that $t_B \sim O(\ln h^{-1})$. On the other hand, from formal expansions of quantum expressions into power series in h it seemed to follow that the breakdown time is generically of the order of, at least, $1/h$ [6]. Recently, numerical studies of various chaotic systems showed that the SCA is able to give quantitatively correct pictures for times well above $\ln h^{-1}$ (see Refs. [7, 8] for the corresponding study of the baker's map). Due to the exponential proliferation of classical trajectories determining the semiclassical evolution, such an explicit comparison is, however, extremely difficult for still larger times.

Substantially more optimistic estimates of t_B were derived recently using general qualitative arguments [9–13]. In [11] it was argued that for generic (two-dimensional) chaotic systems $t_B \sim O(h^{-\alpha})$ with $\alpha = 1/3$. A detailed study of the stadium billiard suggested even $\alpha \sim 1/2$ [12]. Finally, in the case of the baker's map, similar considerations gave a certain indication for a *linear* (in $1/h$) breakdown time [13]. One should note that these conclusions — in view of the absence of direct calculations for such long times — are based on certain assumptions about the "scenario" of the breakdown. In particular, it is assumed that the breakdown is caused by those classical trajectories which enter during their evolution into "dangerous" zones of phase space like the vicinities of caustics or discontinuities, and that the breakdown time is determined by the measure of such "bad" phase-space areas. Although these assumptions give a plausible physical insight into the mechanism of breakdown, one can hardly expect them to give quantitative predictions.

In view of these circumstances, it seems to be very useful to follow the SCA, in a model system, explicitly up to the time when it begins to fail. This turns out to be possible after rewriting sums over periodic orbits in terms of phase-space integrals, the calculation of which requires much less computational effort. (The basic concepts of this approach can be found in Bogomolny's work [14]; see also [15].) In this Rapid Communication, we develop such a phase-space integral formulation for a prototypical hyperbolic system, the baker's map [16–18], and use it to calculate the SCA to the traces of powers of the quantum one-step time-evolution operator U. These traces are of special interest, as they can be used to directly determine the eigenvalues of U. Possible deviations of these eigenvalues from the unit circle signal a loss of unitarity, and provide a basic indication for the failure of the SCA. Our numerical results show that the semiclassical traces start to diverge *exponentially* from their quantum counterparts for times of the order of $h^{-1/2}$.

The baker's map is one of the simplest chaotic systems [16, 17]. It is defined as a mapping of the unit square onto itself,

$$x_{j+1} = \{2x_j\} \ , \ \ y_{j+1} = \frac{1}{2}y_j + \frac{1}{2}[2x_j], \tag{1}$$

where $[x]$ and $\{x\}$ stand for the integer and fractional parts of the coordinate x, respectively. Every point (x, y) on the square is hyperbolic, with the stable and unstable manifolds being parallel to the y and x axes, correspondingly, and has the same Lyapunov exponent of

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$\lambda = \ln 2$. The map possesses two distinct discrete symmetries, $T : (x, y) \to (y, x)$ and $R : (x, y) \to (1-x, 1-y)$. It was shown in [16, 17] that the baker's map has a simple Markov partition comprising the left and right halves of the square. This enables one to uniquely label each trajectory by a doubly infinite binary code $\{\ldots \nu_{-1} \cdot \nu_0 \nu_1 \ldots\}$ in which the jth digit ν_j specifies whether, at the jth iteration of the map, the particle is in the left (0) or right (1) half of the square. The operation of the map is then given by a Bernoulli shift along this code. n-periodic points are points whose symbolic code is n periodic; their coordinates and momenta can be written as rational binary fractions: $x = 0.\overline{\nu_0 \nu_1 \cdots \nu_{n-1}} = \nu/(2^n - 1)$; $y = \bar{\nu}/(2^n - 1)$, where $\nu = \sum_{j=0}^{n-1} \nu_j 2^{n-j-1}$; $\bar{\nu} = \sum_{j=0}^{n-1} \nu_j 2^j$. We label each n-periodic point by the corresponding value of ν. Note that a periodic point is completely determined by its coordinate x.

The *action* of a periodic orbit going through ν was originally derived in [18] and can be expressed as

$$S_\nu = \frac{\nu \bar{\nu}}{2^n - 1} - I_\nu \, , \tag{2}$$

where I_ν is an integer-valued function. Expressing ν in terms of x and using a particular choice of I_ν, we get [19]

$$S_\nu = \sum_{j=0}^{n-1} \nu_j x_j - \sum_{j=0}^{n-1} \nu_j = \sum_{j=0}^{n-1} (x_j - 1)[2x_j] \, . \tag{3}$$

The action (3) is strictly invariant under both R and T symmetries, as well as under iterations of the map. Moreover, S_ν depends exclusively on the coordinates. [One can show that the action recently given in [20] results in an expression similar to (3), with $x_j - 1$ replaced by x_j.] The *quantization* of the baker's map proceeds by constructing a unitary one-step evolution operator U [17, 7, 21]. A representation of U which preserves all symmetries of the classical map in a Hilbert space of dimension L is given by [7] $U_L = G_L^{-1} \times \mathrm{diag}(G_{L/2}, G_{L/2})$, where

$$(G_L)_{mn} = L^{-1/2} \exp\left[-2\pi i(m + 1/2)(n + 1/2)/L\right];$$

$m, n = 0, 1, \ldots, L-1$. This quantization is possible only if $L(= 1/h)$ is a positive even integer.

We turn now to the quantity we are mainly interested in, namely, the SCA of the trace of the n-step time-evolution operator. For chaotic maps, this quantity can be generally represented as a sum over all periodic points of period n [22, 23]:

$$\mathrm{Tr} U_L^n \approx t_L^{(n)} \, , \tag{4}$$

where

$$t_L^{(n)} = \sum_\nu |\det(M_\nu - I)|^{-\frac{1}{2}} e^{i(S_\nu/\hbar - \frac{1}{2}\pi\mu_\nu)} . \tag{5}$$

Here, M_ν, S_ν, and μ_ν are the monodromy matrix, the action, and the Maslov index of the νth n-periodic point, respectively. For the baker's map, $t_L^{(n)}$ was considered in detail in Refs. [18, 8] and is given by

$$t_L^{(n)} = \frac{1}{2\sinh(n \ln 2/2)} \sum_\nu e^{2\pi i L S_\nu} \, , \tag{6}$$

where the sum runs over all periodic points of period n. As was noticed in Ref. [18], special care must be taken for the fixed points $(0, 0)$ and $(1, 1)$ lying at the *corner* generated in phase space by the chopping procedure underlying the baker's transformation. For these points the stationary phase approximation is invalid, and one can show that they enter the semiclassical traces with an *enhanced* amplitude (as compared to other periodic points), the enhancement factor depending logarithmically on h [20]. We do not go into this discussion, since our main aim in this Rapid Communication is the behavior of $t_L^{(n)}$ for *large* n, and the relative error caused by the improper treatment of a single periodic point is of order 2^{-n}, for any fixed h.

In order to construct a semiclassical transfer operator for the calculation of $t_L^{(n)}$, we make use of the particular structure of the baker's map which allowed us to express, both on the classical and on the semiclassical level, all relevant quantities in terms of the orbit coordinates only. Let us define the one-dimensional mapping $B : x \to x' = \{2x\}$; $0 \le x < 1$. From the discussion below Eq. (1) it follows that the n-periodic points of this mapping coincide with the coordinates of the n-periodic points of the original baker's map. This implies that any sum over periodic points of the baker's map with momentum-independent weight factors can be written as a sum over periodic points of the reduced map B. In particular, this applies to the expression (6) for $t_L^{(n)}$. A semiclassical transfer operator can then be obtained by generalizing the classical Ruelle-Perron-Frobenius operator $U_{\mathrm{cl}}(x, x')$ [24] defined as the integral kernel transforming an initial probability distribution $\rho_0(x)$ over phase space into the corresponding distribution after one application of the map:

$$\rho_1(x') = \int_0^1 U_{\mathrm{cl}}(x, x') \rho_0(x) dx \, . \tag{7}$$

From the structure of the mapping B one immediately concludes that

$$U_{\mathrm{cl}}(x, x') = \frac{1}{2}\left[\delta\left(x - \frac{x'}{2}\right) + \delta\left(x - \frac{x' + 1}{2}\right)\right] . \tag{8}$$

Iterating Eq. (7) n times, one obtains the corresponding n-step operator as the nth power of U_{cl}. The traces of these operators can be expressed in terms of the periodic points of the mapping B. Adopting the convention $\int_0^1 \delta(x) dx = 1/2$, we have

$$\mathrm{Tr} U_{\mathrm{cl}}^{(n)} = \frac{1}{2^n - 1} \sum_\nu c_\nu \, , \tag{9}$$

where the sum goes over all n-periodic points and c_ν equals 1 for the inner, and 1/2 for the border points.

Equation (9) expresses a sum over periodic points in terms of the trace of an operator defined over configuration space. This suggests that other sums over periodic points differing from (9) by the weight given to different points can be expressed in terms of configuration space integrals as well. In particular, we wish to derive operators $W_L^{(n)}$, defined in terms of classical dynamics, such that

$$t_L^{(n)} = \mathrm{Tr} W_L^{(n)} \equiv \int_0^1 dx W_L^{(n)}(x,x) \ . \tag{10}$$

Comparing (9) with the semiclassical expression (6), one sees that $t_L^{(n)}$ is given by the right-hand side of Eq. (9) with c_ν replaced by $c_\nu 2^{n/2} \exp(2\pi i L S_\nu)$. Taking into account that the action (3) factorizes into a sum of one-step contributions, we conclude that $W_L^{(n)}$ is given by the nth power of a *one-step operator* $W_L(x,x')$:

$$W_L^{(n)}(x_0,x_n) = [(W_L)^n](x_0,x_n)$$
$$\equiv \int_0^1 dx_1 \cdots \int_0^1 dx_{n-1} \prod_{j=0}^{n-1} W_L(x_j,x_{j+1}) \ , \tag{11}$$

where

$$W_L(x,x') = 2^{-\frac{1}{2}} \left[\delta\left(x - \frac{x'}{2}\right) + \delta\left(x - \frac{x'+1}{2}\right) \right] e^{2\pi i L(x-1)[2x]} \ . \tag{12}$$

Equations (10)–(12) are the main result of the present Rapid Communication. They express the periodic orbit sum (6) for the SCA to the traces of the n-step time-evolution operator of the quantum baker's map in terms of a semiclassical operator defined over configuration space. We emphasize that Eqs. (10)–(12) give an *exact* representation of the semiclassical traces, valid for any number of iterations n. A detailed study of the properties of the operator W_L allows, therefore, for a consideration of the large n regime without explicitly considering a huge number of periodic orbits.

It is convenient to study the operator W_L in the Fourier representation given (for even L) by

$$\tilde{W}_L(m,k) = \int_0^1 dx \int_0^1 dx' \ e^{2\pi i(mx-kx')} \ W_L(x,x')$$
$$= \begin{cases} \frac{1}{\sqrt{2}}, & m = 2k \text{ or } m = 2k - L \\ \frac{iL\sqrt{2}}{\pi(m-2k)(m+L-2k)}, & m \text{ odd} \\ 0 & \text{otherwise} \ . \end{cases} \tag{13}$$

The indices m and k take all integer values from $-\infty$ to ∞.

In order for (13) to be useful, one must be able to efficiently approximate $t_L^{(n)}$ via finite dimensional truncations of $\tilde{W}_L$. For this purpose we truncate $\tilde{W}_L$ to a K-dimensional matrix $\tilde{W}_L^{[K]}$ with indices ranging from $-K/2+1$ to $K/2$. Our numerical results show a fast convergence of the approximations to $t_L^{(n)}$ with enlarging the truncation size: Fig. 1 gives a comparison between the expressions for $t_L^{(n)}$ obtained by directly computing the periodic orbit sum (6) and the traces of the nth power of different approximations to $\tilde{W}_L$, for $L = 200$ and $L = 250$, in the range $1 \le n \le 31$ for which periodic orbit data exist. One sees that, except for very small n, a good approximation is reached already for $K \sim 2L$. This allows us to continue Eq. (10) as $t_L^{(n)} = \mathrm{Tr} W_L^{(n)} = \mathrm{Tr}(\tilde{W}_L)^n = \lim_{K\to\infty} (\tilde{W}_L^{[K]})^n = \lim_{K\to\infty} \sum_{j=1}^K [e_j(L,K)]^n$, where the $e_j(L,K)$ are the eigenvalues of the matrix $\tilde{W}_L^{[K]}$.

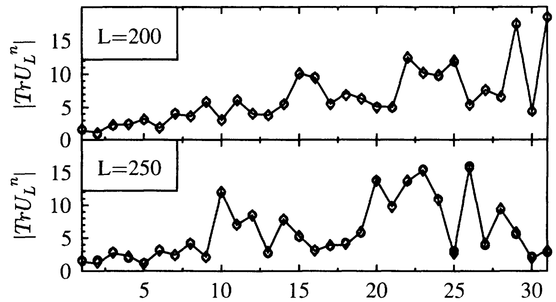

FIG. 1. Comparison between $|t_L^{(n)}|$ and $|\mathrm{Tr}(\tilde{W}_L^{[K]})^n|$ for $K = 512$ (circles) and $K = 1024$ (diamonds).

A typical spectrum of $\tilde{W}_L^{[K]}$ is shown in Fig. 2 ($L = 100, K = 256$) and compared to the spectrum of the quantum mechanical operator U_L. U_L is a $L \times L$ unitary matrix, and therefore its L eigenvalues all lie on the unit circle. The semiclassical spectrum shares this gross feature: One clearly sees that the number of eigenvalues of $\tilde{W}_L^{[K]}$ with absolute values ≈ 1 is of the order L. This result is highly gratifying, since there seems to be no *a priori* reason for the nonunitary matrix $\tilde{W}_L$ (and its finite dimensional truncations) to behave in such a way. It shows that the SCA is in a certain sense "close" to quantum mechanics. However, the step function characteristic of the quantum spectrum is smeared out at both sides: There are eigenvalues with absolute values larger than 1, and there is a long tail of small eigenvalues. Note that for the long-time behavior we are interested in, only the eigenvalues in the vicinity of the unit circle are of relevance. We investigated carefully the dependence of these large eigenvalues on the truncation size K and found that their values stabilize very rapidly: The values obtained for matrix sizes $K \sim 2L$ are typically changed by a further increase of K by not more than $10^{-4} - 10^{-3}$ [19].

We find, therefore, that the effort involved in computing $t_L^{(n)}$ is only $O(L^3)$, i.e., similar to the computational load of the corresponding quantum calculation — in contrast to the exponential rise in complexity associated with explicit summation over periodic orbits. This enables us to go far beyond the times which would be treatable by a straightforward periodic orbit calculation, and to investigate the long-time accuracy of the SCA without any further assumptions. For this purpose we return to Fig. 2: It shows that the leading semiclassical

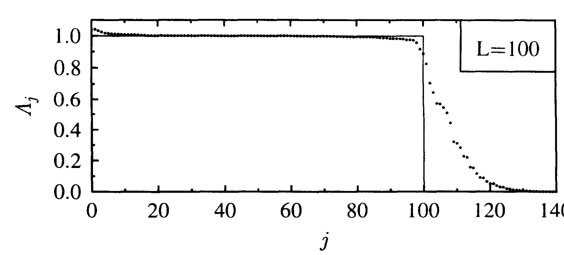

FIG. 2. $\Lambda_j \equiv |e_j(L,K)|$ for $L = 100$ and $K = 256$ in comparison with the quantum spectrum (full line).

eigenvalues do not lie *exactly* on the unit circle, but at some distance from it. This reflects the fact that the SCA violates, in general, the unitarity of the time evolution. As an immediate consequence, the semiclassical evolution will be determined, after some initial time, by the eigenvalue with the *largest* absolute value (note that this leading eigenvalue is not degenerate in absolute value). Let us denote this value by $1 + \Delta_L$. Then the long-time behavior of $t_L^{(n)}$ is given by $|t_L^n| \sim (1+\Delta_L)^n \approx e^{\Delta_L n}$. Since the magnitude of the trace of any power of the quantum mechanical propagator is bounded by L, the SCA turns out to be meaningful, for a given value of L, at most up to a number of steps n which is roughly given by the inverse of Δ_L, whereas for larger times the traces $t_L^{(n)}$ will increase exponentially (and so will all quantities expressed in terms of it, e.g., the semiclassical spectral form factor [25]). We studied the L dependence of Δ_L numerically over the ranges $0 < L < 500$ and $990 < L < 1000$ (see Fig. 3). The data suggest a roughly powerlike law, the best fit depending on the range of data included, and varying between $\Delta_L \sim L^{-0.46}$ and $\Delta_L \sim L^{-0.63}$.

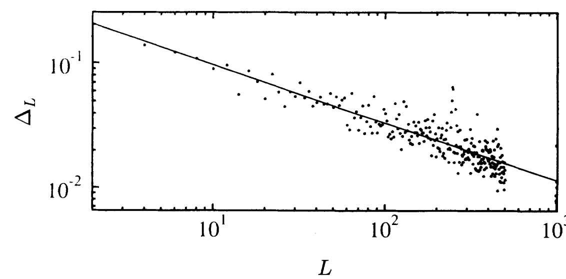

FIG. 3. Distance Δ_L of the leading eigenvalue of $\tilde{W}_L^{[K]}$ ($K = 1024$) from the unit circle as a function of L. The straight line is the best powerlike fit through all data points and is given by $\Delta_L = 0.29L^{-0.47}$.

We conclude that, for the baker's map, the SCA to the traces of the time-evolution operator diverge exponentially from the exact quantum results for times $t_B \sim O(h^{-\alpha})$ with $\alpha \approx 1/2$. This confirms the general expectation of a powerlike behavior [9–13]. It shows, however, that the *linear* dependence of t_B on $1/h$ which would be necessary for describing individual quasienergies in the limit $h \to 0$ is not reached. An important question is, therefore, whether this fact is specific to the baker's map. Indeed, there seems to be a connection between the $h^{-1/2}$ dependence of t_B, and the discontinuous form of the map: Investigating the optical realization of the baker's map proposed recently [26], one sees that the discontinuity of the classical map implies diffraction effects in the quantum case which are known to give contributions of order $h^{-1/2}$ to semiclassical quantities not covered by periodic orbit expressions [21]. One could expect, therefore, that for smooth systems the SCA would not suffer from this peculiarity, and would have an extended domain of validity. Comparing our findings to those claimed by O'Connor *et al.* [13], we see that the results of argumentation based on considerations of certain areas of phase space should be considered with caution. Having at our disposal a method for directly calculating the long-time behavior of chaotic systems in the semiclassical approximation will enable us to test many of the ideas about the origin of the breakdown of the SCA in more detail. We are hopeful that the method which turned out to be so effective for the baker's map can be of help in analyzing more complicated chaotic systems as well.

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